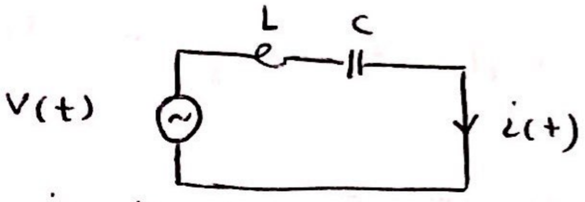
① What is resonant? $\begin{cases} \text{Amp. over time} \\ \text{Freq. dependence} \end{cases}$

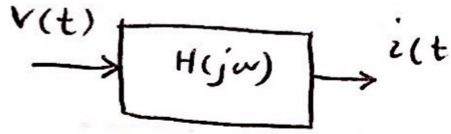
② Example of resonant system (Series LC)

↳ How to find if it is resonant? Excite @ all freqs and see which one is selected.

↳ Excite @ oscillating freq. and see amplification.

↳ $\boxed{\text{The transfer function of the resonant system.}}$



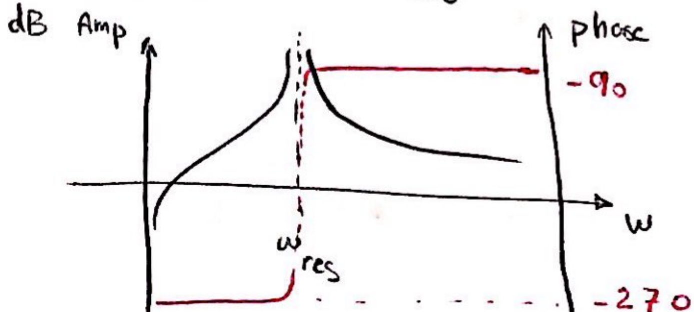


in time-domain analysis

- We saw upon excitation at resonance (i.e., $\omega_{res} = \frac{1}{\sqrt{LC}}$) the oscillating response grows with time:
  (i.e., when $v(t) = \sin(\omega_{res} t)u(t)$, then $i(t) = \frac{L}{2} t \sin(\omega_{res} t)u(t)$)

- Now let's find the transfer function of this resonant system:

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + \frac{1}{Cs}} \rightarrow H(j\omega) = \frac{1}{j\omega L + \frac{1}{j\omega C}} = \frac{-j\,\omega/L}{\omega^2 - \frac{1}{LC}}$$



* Resonance is a peak @ ω_{res}!

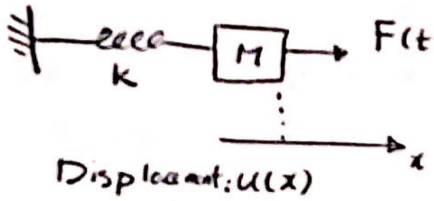
* The phase changes by 180° @ ω_{res}!

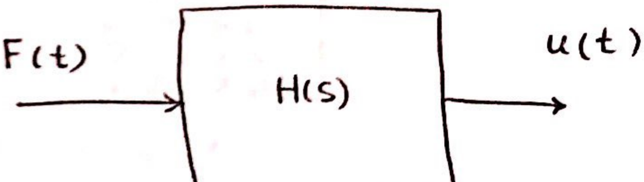
* Generalization: Any system with $H(s) = \frac{N(s)}{D(s)}$, where $D(s)$ has a complex-conjugate pole pair of $\pm j\omega_0$ is resonant, with a resonance freq. of ω_0.

HW2, P1: Prove this statement by time-domain analysis. (Hint: excite the system @ ω_0, and show the amplification happens!)

↳ Immediate extension: if $H(s)$ has K complex-conjugate poles at $\pm j\omega_i$ ($i = 1$ to K), then the system has K resonance modes!

Example 2: Mechanical resonant system (harmonic oscillator):





Newton's 2nd law

$$\underbrace{M\frac{d^2u}{dt^2}}_{} + \underbrace{ku}_{\text{Hooke's law}} = F \xrightarrow{\mathcal{L}} U(s)(Ms^2 + K) = F(s)$$

$$\rightarrow H(s) = \frac{U(s)}{F(s)} = \frac{1}{Ms^2 + K}$$

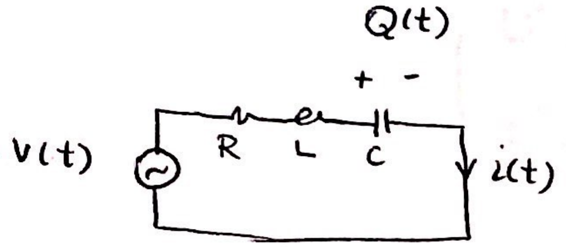
→ poles: $s = \pm j\sqrt{\frac{K}{M}}$ → A resonant system; $\omega_{res} = \sqrt{\frac{K}{M}}$.

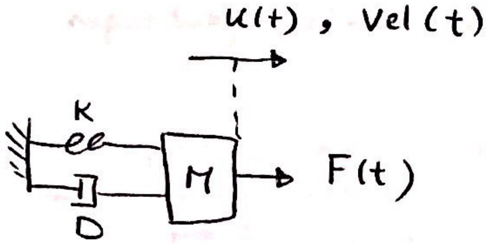
HW2, p2: We want to make an actuator with largest possible displacement @ lowest input force! What is your solution?

Damped oscillation / resonance: In reality, all resonant systems have at least one damping mechanism that attenuates the signal.

- ↳ Electrical: resistance, elec-mag radiation, etc.
- ↳ Mechanical: air damping, thermal noise, friction, etc.

Example 3:





$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

$$M\frac{d^2u}{dt^2} + D\frac{du}{dt} + Ku = F(t)$$

L	M
R	D
C	$1/K$
V	F
Q	u
i	vel

* Resonant systems ~~while~~ are "dynamically equivalen"!
  - ↳ All the math interchangable!
  - ↳ Note: loss/attenuation in both are essentially same $\left\{\begin{matrix} R \\ \& \\ D \end{matrix}\right.$

* Now time & freq. analysis in pressence of loss/damping.
  - ↳ Assuming impulse input & finding $u(t)$ {or equivalently $Q(t)$}.

$$\xrightarrow{\mathcal{L}} \begin{cases} \left(Ls^2 + Rs + \frac{1}{C}\right) Q(s) = 1 \rightarrow Q(s) = \dfrac{1}{Ls^2 + Rs + 1/C} \\ (Ms^2 + Ds + K)\, U(s) = 1 \rightarrow U(s) = \dfrac{1}{Ms^2 + Ds + K} \end{cases}$$

* Poles of $V(s)$: $\dfrac{-D \pm \sqrt{D^2 - 4MK}}{2M}$

→ As we know from circuits 1/2 or ODE : three senarios could happen:

$$\begin{cases} D^2 > 4MK \rightarrow \text{two real-roots} \rightarrow \text{over-damped.} \\ D^2 = 4MK \rightarrow \text{repeating real-roots} \rightarrow \text{critically damped.} \\ D^2 < 4MK \rightarrow \text{two complex-conjugate roots! underdamped.} \end{cases}$$

→ This is the required criteria for resonate system!

* Assuming under-damped case :

→ poles are $-\dfrac{D}{2M} \pm j\left(\dfrac{K}{M} - \dfrac{D^2}{4M^2}\right)^{0.5}$

Damping ratio.

* Defining : $\begin{cases} \zeta = \dfrac{D}{2\sqrt{KM}} \\ \omega_0 = \sqrt{\dfrac{K}{M}} \end{cases}$ → poles : $\boxed{-\zeta\omega_0 \pm j\,\omega_0\sqrt{1-\zeta^2}}$

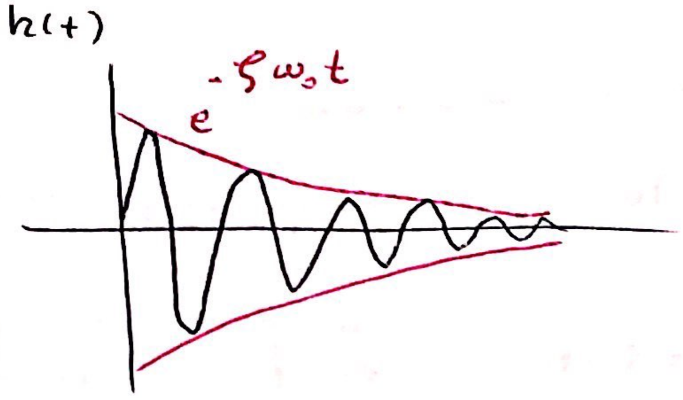
(undamped) Natural resonence freq of the system

$\omega_d \triangleq \omega_0\sqrt{1-\zeta^2}$
Damped res. freq.

$$\Rightarrow \quad u(t) = \sqrt{\frac{1}{KM}}\, e^{-\zeta\omega_0 t} \sin(\omega_d t + \varphi)$$

Key take-aways: $\begin{cases} \text{1) Damping : exponentially attenuates resonance oscilation} \\ \text{2) Damping : reduces resonance freq to } \omega_d\text{ !} \end{cases}$

* Plotting impulse response: $u(t) = \frac{1}{\sqrt{KM}} e^{-\zeta\omega_0 t} \sin(\omega_d t + \varphi)$:



* Quality factor : The lower the damping rate, the higher the quality of resonator!

$$\boxed{\underset{\text{(The quality factor)}}{Q} \triangleq 2\pi \frac{\text{Energy stored in the system}}{\text{Energy lost per cycle}}}$$

* For the harmonic oscillator, energy stored in the system at the beginning of each $\underset{\text{i.e. } NT}{\text{cycle}}$ is : $\frac{1}{2} K \left\{ \frac{1}{\sqrt{KM}} e^{-\zeta\omega_d NT} \right\}^2$

HW 2, P3 : why the above statement is correct?

$$\Rightarrow Q = \frac{\frac{1}{2M} e^{-2\zeta\omega_d NT} \cdot 2\pi}{\frac{1}{2M} e^{-2\zeta\omega_d NT} - \frac{1}{2M} e^{-2\zeta\omega_d (N+1)T}} = \frac{2\pi}{1 - e^{-2\zeta\omega_d T}} \simeq \frac{2\pi}{2\zeta\omega_d T}$$

$$\Rightarrow Q = \frac{2\pi}{2\zeta \underbrace{\boxed{\omega_d T}}_{=2\pi}} = \frac{1}{2\zeta}$$

$$\boxed{Q = \frac{1}{2\zeta}}$$

HW2, p 31: Derive the time-domain wave-form of damped resonant system for $\sin(\omega_d t)$ excitation!

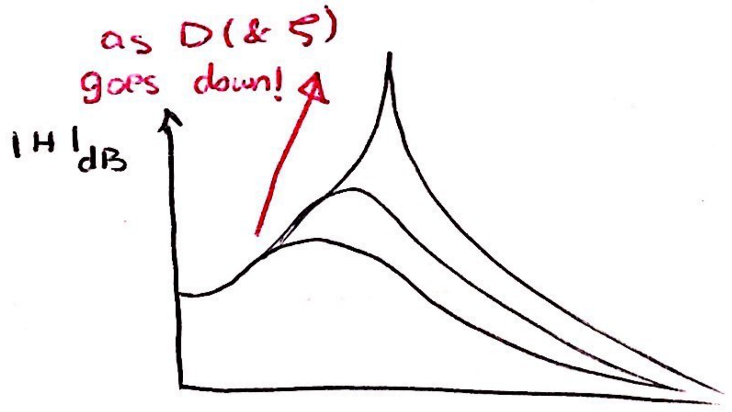
---

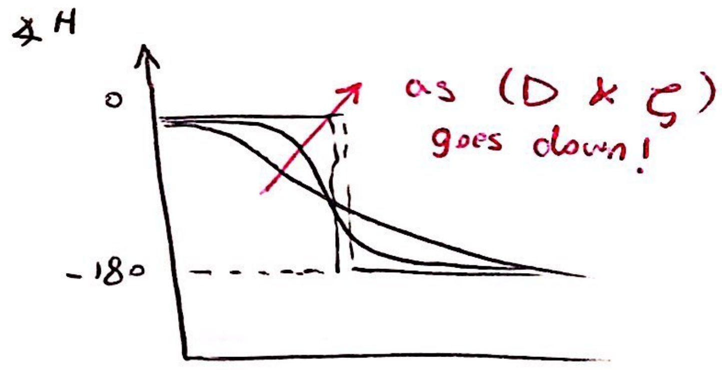
* Now freq.-domain analysis of damped-resonant system:

$F(t) \longrightarrow \boxed{H(s)} \longrightarrow u(t)$

$$H(s) = \frac{1}{Ms + R + \frac{1}{Cs}}$$

$$\Rightarrow \boxed{H(j\omega) = \frac{-1/M}{(\omega^2 - \omega_0^2) - j(2\zeta\omega_0\omega)}}$$





* From Freq. response, the lower D & ζ the sharper the peak & 180° phase transition!! [We can use the sharpness of the peak as a measure of resonator quality!]

Sharpness of the peak: $\dfrac{\omega_{res}}{\Delta\omega_{-3dB}}$; $\Delta\omega_{-3dB} = \omega_{-3dB,H} - \omega_{-3dB,L}$

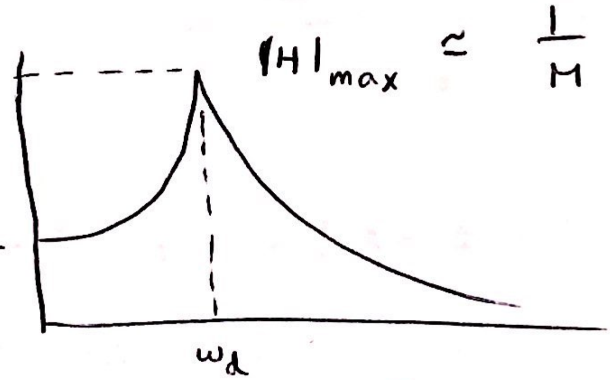
* Let's find/calculate $\frac{\omega_{res}}{\Delta\omega_{-3dB}}$:

@ ω_{-3dB} : $|H(j\omega_{-3dB})| = \frac{1}{\sqrt{2}} |H(j\omega_d)| \simeq \frac{1}{\sqrt{2}} \frac{+\frac{1}{M}}{\omega_0^2\xi^2 - j2\xi\omega_0\omega_d}$

$$\Rightarrow \frac{\left|(\omega_{-3dB}^2 - \omega_0^2) - j(2\xi\omega_0\omega_{-3dB})\right|}{\sqrt{2}} = \left|\omega_0^2\xi^2 - j2\xi\omega_0\omega_d\right|$$

HW2, Bonus (30 points): solve the eq. above to find $\omega_{-3dB,H}$ & $\omega_{-3dB,L}$. show that

$$\boxed{\frac{\omega_d}{\Delta\omega_{-3dB}} = \frac{\omega_d}{\omega_{-3dB,H} - \omega_{-3dB,L}} = Q}$$



$|H|_{max} \simeq \frac{1}{M} \cdot \frac{1}{2\xi\omega_0^2} = \frac{1}{2\xi \cdot K} = \boxed{\frac{Q}{K}}$

$|H(\omega=0)| = \frac{1/M}{\omega_0^2} = \boxed{\frac{1}{K}}$

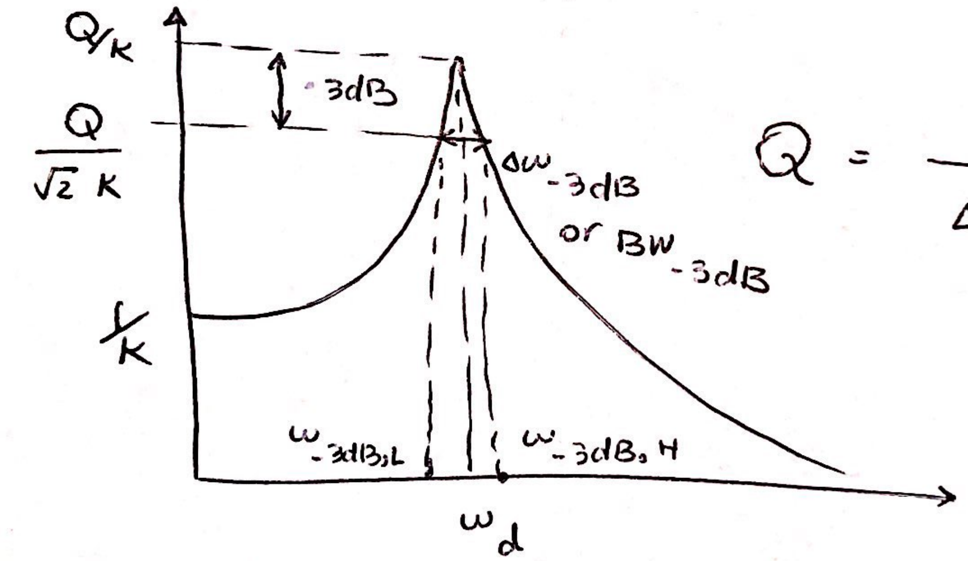
* At resonance amp of displacement is Q times that of DC!

Conclusion of freq. domain analysis:

* The resonance shows as a peak @ $\omega_d = \omega_0 (1-\zeta^2)^{0.5}$
* The sharpness of the peak is $Q = \dfrac{\omega_d}{\Delta\omega_{-3dB}} = \dfrac{1}{2\zeta}$
* For $\dfrac{U(j\omega)}{F(j\omega)}$, the oscillation amplitude at ω_{res}

is Q times displacement @ $\omega = 0$!

$$H(j\omega)_{dB} = \left(\frac{U(j\omega)}{F(j\omega)}\right)_{dB}$$



$$Q = \frac{\omega_d}{\Delta\omega_{-3dB}}$$

HW2, Bonus (30 points). Prove Q can be extracted from $H(j\omega)$ phase slope at ω_{res}:

i.e., $Q = \dfrac{\omega_{res}}{2} \cdot \dfrac{d\measuredangle H(j\omega)}{d\omega}$