Lecture 3:

- Why Mechanical Resonator? Why Micro-Scale?
- Electrical Modeling of MEMS Resonators

Prof. Roozbeh Tabrizian
Department of Electrical and Computer Engineering
University of Florida
URL: http://phonon.ece.ufl.com; Email: rtabrizian@ufl.edu
Discrete Electrical (RLC) vs Mechanical Resonators?

- What metrics should be used for comparison?
  - $Q$, frequency, integrability in an electronic system, size, cost

- On-chip inductor + capacitor (RLC electrical resonator)
- Quartz crystal resonator (Mechanical – Piezo transducer)
- Silicon MEMS resonator (Mechanical – Electrostatic transducer)
Q Comparison

- Integrated LC resonators provide very low $Q$ (<20)!
  - Due to the loss of lines used to create inductors!
- Mechanical resonators provide $Q$ as high as 10s of millions!

$$Q = \frac{\omega_{\text{res}}L}{R} \quad (= \text{over } 1 - 20)$$

$$Q = \frac{\omega_{\text{res}}M}{D} \quad (= \text{over } 10^3 - 10^6)$$
**Q Comparison**

- Higher $Q$ means:
  - lower energy dissipation and power consumption
  - Higher frequency selectivity
  - Higher signal to noise ratio
Frequency Scaling

- MEMS resonators benefit from high operation frequency
  - Due to micro-scale size
  - Smaller mass
  - Larger spring constant

\[
f_{\text{res}} = \frac{1}{2\pi} \sqrt{\frac{K}{M}} \quad (= \text{over } 10^3 \text{ -- } 10^{10}\text{Hz})
\]
Lumped Element Mechanical Resonator

- Creating micro-scale spring and mass by patterning silicon
  - Patterning microscale mass and spring (lithography & etching)
  - Releasing structure to enable free vibration

1) Silicon-on-insulator substrate
2) Patterning Silicon device-layer (lithography & etching)
3) Selective etching of SiO$_2$ to release the resonator
Lumped Element Mechanical Resonator

- The capacitor (i.e., transducer) enables applying mechanical force through application of voltage (transducing voltage to force)
  - To enable using mechanical resonator as an electrical component!

\[
\begin{align*}
&v_{ac} + V_{DC} \\
&K \\
&D \\
&g
\end{align*}
\]

Capacitive transducer: converting voltage to force!
The First-Ever MEMS: The Resonant Gate Transistor

- The first-ever invention on MEMS: 1968, Nathanson et. al.
  - At resonance of the cantilever-gate, the vibration amplitude is $Q$-times larger.
  - At resonance, the resistance of channel between source and drain reduces.
  - This creates a transistor with best channel conduction at resonance frequency!

![Diagram of Resonant Gate Transistor]
Lumped Element Spring

-Benefiting from elasticity of semiconductor / insulators

\[ K_Y \approx E \cdot \frac{H \cdot W^3}{4L^3} \]

\[ K_z \approx E \cdot \frac{W \cdot H^3}{4L^3} \]

\[ K_X \approx E \cdot \frac{H \cdot W}{L} = \frac{E \cdot A}{L} \]
Lumped Element Spring: Numerical Example

- Find spring values assuming $L = 300\mu m$, $H = 40\mu m$, $W = 40\mu m$, $E = 165$ GPa

$$K_{flex} \cong 165 \times 10^9 \cdot \frac{40 \times 10^{-6} \times (40 \times 10^{-6})^3}{4 \times (300 \times 10^{-6})^3} = 3911 \text{ N/m}$$

$$K_{tens} \cong 165 \times 10^9 \cdot \frac{40 \times 10^{-6} \times 40 \times 10^{-6}}{300 \times 10^{-6}} = 2933 \times 10^6 \text{ N/m}$$
Lumped MEMS Resonator: Numerical Example

- Find resonance frequency assuming $L_{\text{mass}} = 200 \mu\text{m}$, $\rho = 2330 \text{ kg/m}^3$

\[
M = 2330 \times 40 \times 10^{-6} \times (200 \times 10^{-6})^3 = 3.728 \times 10^{-9} \text{ kg}
\]

\[
f_{\text{res}} = \frac{1}{2\pi} \sqrt{\frac{K_{\text{flex}}}{M}} = 163 \text{ kHz}
\]

\[
f_{\text{tensile}} \approx 1000 \times f_{\text{flex}}
\]

\[
M = 2330 \times 40 \times 10^{-6} \times (200 \times 10^{-6})^3 = 3.728 \times 10^{-9} \text{ kg}
\]

\[
f_{\text{res}} = \frac{1}{2\pi} \sqrt{\frac{K_{\text{tensile}}}{M}} = 141 \text{ MHz}
\]
Effect of Distributed Mass

• In practice, the mechanical spring has mass
  - Loads (i.e., reduces) the resonance frequency

• The majority of MEMS resonators are distributed (No lumped M & K)
  - Displacement variable $U$ is a function of $(x, y, z)$ and $t$
  - $U$ is called the mode-shape function

\[
 f_{res} = \frac{1}{2\pi} \sqrt{\frac{K}{M + M_{spring}}} 
\]

\[
 \bar{U}(x, y, z, t) = U_0 \sin(\omega_{res} t) 
\]

\[
 \bar{U}(x, y, z, t) = \begin{bmatrix} U_x(x, y, z) \\ U_y(x, y, z) \\ U_z(x, y, z) \end{bmatrix} \sin(\omega_{res} t) 
\]
**MEMS Resonance Mode-Shape**

- The spatial displacement vector \( \begin{bmatrix} U_x(x, y, z) \\ U_y(x, y, z) \\ U_z(x, y, z) \end{bmatrix} \) is called mode-shape function.
  - Since it formulates the displacement pattern of the structure.

- Example: Extensional beam

\[
\begin{bmatrix}
0 \\
sin(y\pi/W_0) \\
0
\end{bmatrix}
\]
Electrical Equivalent Model of MEMS Resonator

- Any MEMS resonator is composed from:
  1. Micro-mechanical resonator
  2. Electro-Mechanical transducer

- To use MEMS Resonator in electronic systems an electrical equivalent model is desired

- Electromechanical transducers are represented through transformers
  - To generate force for resonance actuation
  - To sense displacement / velocity

- Mechanical resonator should be modeled using R,L,C components
Lumped Element Modeling of MEMS Resonator

1. Model the micro-mechanical resonator:
   - Find the M-K-D harmonic oscillator with the same energy dynamics
   - Show the M-K-D system using electrical equivalent symbol (series resonator)

2. Model the electro-mechanical transducer
   - Represented as a lumped transformer converting voltage / current to force / velocity and vice versa
   - Identify the electrical parasitic elements
Lumped Element Mechanical Model for Resonator

- Extracting equivalent M-K-D for a mechanical resonant system
  - In distributed systems equivalent M-K-D should be derived
- The LEM has the same $f_{res}$, $Q$ and overall stored energy
- The LEM model depends on observing point on the structure
  - Where we embed the transducer / apply the force (& usually has the largest displacement or tension)

$$F(t) = U_{observer} \sin(\omega t)$$
Lumped Element Model for Resonant System

• Energy method can be used to find $M_{eq}$ and $K_{eq}$

$$E_{overall} = E_{kinetic}(t) + E_{potential}(t) = \frac{1}{2}M \left(\frac{dU}{dt}\right)^2 + \frac{1}{2}K U^2$$

$$= \frac{1}{2} M \omega_0^2 U_0^2 (\cos(\omega_0 t))^2 + \frac{1}{2} KU_0^2 (\sin(\omega_0 t))^2$$

$$= \frac{1}{2} KU_0^2 = \frac{1}{2} M \omega_0^2 U_0^2$$

$$\omega_0 = \sqrt{\frac{K}{M}}$$

$$U(t) = U_0 \sin(\sqrt{\frac{K}{M}} t)$$

$$E_{overall} = E_{kinetic,max} = E_{potential,max}$$
Lumped Element Model for Resonant System

- Using the maximum kinetic and potential energy stored in the distributed system to extract $K_{eq}$ and $M_{eq}$, respectively.

\[ E_{\text{kinetic, max}} \]

\[ E_{\text{potential, max}} \]

\[ E_{\text{stored}} = \frac{1}{2} M_{eq} U_{0, \text{observer}}^2 \omega_0^2 \]

\[ E_{\text{stored}} = \frac{1}{2} K_{eq} U_{0, \text{observer}}^2 \]
Lumped Element Model for Resonant System

- To calculate the volumetric integral, we just need the mode-shape function

\[ E_{\text{potential,max}} \]

\[ E_{\text{stored}} = \iiint E_{\text{potential}}(x, y, z) \, dV = \frac{1}{2} M_{eq} U_{0,\text{observer}}^2 \omega_0^2 \]

\[ E_{\text{kinetic,max}} \]

\[ E_{\text{stored}} = \iiint E_{\text{kinetic}}(x, y, z) \, dV = \frac{1}{2} K_{eq} U_{0,\text{observer}}^2 \]
Distributed System Case Study: Extensional Beam

- Derive the lumped element mass / spring equivalents for the extensional beam
  - Having the mode shape function $U(x, y, z) = U_0 \cdot \sin\left(\frac{\pi}{W} \cdot x\right)$
  - Mode shape is independent of $y$ and $z$: planar movement
  - Transducers will be embedded at the ends of the beam

Cross section area $= A$
Distributed System Case Study: Extensional Beam

\[ \Delta M = \rho \cdot \Delta Volume = \rho \cdot A \cdot \Delta x \]

\[ \Delta K = \frac{E \cdot A}{\Delta x} \]

\[ \Delta E_{overall}(x) = \Delta E_{kinetic}(x, t) + \Delta E_{potential}(x, t) \]
\[ = \Delta E_{kinetic,max}(x) = \Delta E_{potential,max}(x) \]
Distributed System Case Study: Extensional Beam – $M_{eq}$ Extraction

\[ \Delta E_{kinetic}(x, t) = \frac{1}{2} \Delta M \cdot \text{vel}(x, t)^2 \]
\[ v(x, t) = \frac{\partial u(x, t)}{\partial t} = Y(x) \cdot \omega_0 \cdot \cos(\omega_0 t) \]

\[ \Delta E_{overall}(x) = \Delta E_{kinetic, max}(x) = \frac{1}{2} \Delta M \cdot \{\text{vel}(x, t)\}_{max}^2 \]
\[ = \frac{1}{2} \Delta M \cdot \{U(x) \cdot \omega_0\}^2 \]
\[ = \frac{1}{2} \cdot \rho \cdot A \cdot \Delta x \cdot \{U(x) \cdot \omega_0\}^2 \]

\[ E_{overall} = \int_{-W/2}^{W/2} \Delta E_{overall}(x) = \int_{-W/2}^{W/2} \frac{1}{2} \cdot \{U(x) \cdot \omega_0\}^2 \cdot \rho \cdot A \cdot dx \]
\[ = \frac{1}{4} \rho \cdot A \cdot W \cdot U_0^2 \cdot \omega_0^2 = \frac{1}{2} M_{eq} \cdot U_0^2 \cdot \omega_0^2 \]

\[ M_{eq} = \int_{-W/2}^{W/2} \frac{1}{2} \cdot \left\{ \frac{U(x)}{U_0} \right\}^2 \cdot \rho \cdot A \cdot dx = \rho \cdot A \cdot W \cdot \frac{1}{2} = \frac{M_{overall}}{2} \]
Distributed System Case Study: Extensional Beam – $K_{eq}$ Extraction

$$\Delta E_{potential}(x, t) = \frac{1}{2} \Delta K \cdot \Delta U(x, t)^2$$

$$\Delta u(x, t) = U(x + \Delta x, t) - U(x, t) = \frac{\partial U(x, t)}{\partial x} \cdot \Delta x \quad \Delta K = \frac{E \cdot A}{\Delta x}$$

$$\Delta E_{potential}(x, t) = \frac{1}{2} \Delta K \cdot \Delta U(x, t)^2 = \frac{1}{2} \cdot \frac{E \cdot A}{\Delta x} \cdot \left( \frac{\partial U(x, t)}{\partial x} \cdot \Delta x \right)^2 = \frac{1}{2} \cdot \frac{E \cdot A}{\Delta x} \cdot \left( \frac{\partial U(x, t)}{\partial x} \right)^2 \cdot \Delta x$$

$$\Delta E_{overall}(x) = \Delta E_{potential, max}(x) = \frac{1}{2} \cdot \frac{E \cdot A}{\Delta x} \cdot \left( \frac{\partial U(x, t)}{\partial x} \right)_{max}^2 \cdot \Delta x$$

$$\frac{\partial U(x, t)}{\partial x} = \sin(\omega_0 t) \cdot \frac{\partial U(x)}{\partial x}$$
Distributed System Case Study: Extensional Beam – $K_{eq}$ Extraction

\[
\Delta E_{overall}(x) = \frac{1}{2} \cdot E \cdot A \cdot \left(\frac{\partial U(x)}{\partial x}\right)^2 \cdot \Delta x = \frac{1}{2} \cdot E \cdot A \cdot \left\{U_0 \cdot \frac{\pi}{W} \cdot \cos\left(\frac{\pi}{W} \cdot x\right)\right\}^2 \cdot \Delta x
\]

\[
E_{overall} = \int_{-W/2}^{W/2} \Delta E_{overall}(x) = \int_{-W/2}^{W/2} \frac{1}{2} \cdot E \cdot A \cdot \left\{U_0 \cdot \frac{\pi}{W} \cdot \cos\left(\frac{\pi}{W} \cdot x\right)\right\}^2 \cdot dx = \frac{E \cdot A \cdot \pi^2 \cdot U_0^2}{4W} = \frac{1}{2} K_{eq} \cdot U_0^2
\]

\[
K_{eq} = \frac{E \cdot A \cdot \pi^2}{2W}
\]
Distributed System Case Study: Extensional Beam – $f_0$ Extraction

\[
f_0 = \frac{1}{2\pi} \omega_0 = \frac{1}{2\pi} \cdot \sqrt{\frac{K_{eq}}{M_{eq}}} = \frac{1}{2\pi} \cdot \sqrt{\frac{E \cdot A \cdot \pi^2}{\rho \cdot A \cdot W}} = \frac{1}{2W} \cdot \sqrt{\frac{E}{\rho}}
\]
Electrical Representation of M-K-D

- Similar ODEs representing the two system: suggests a simple representation of M-K-D using R-L-C

\[
M \frac{d^2 U}{dt^2} + D \frac{dU}{dt} + KU = F
\]

\[
Q = U
\]

\[
I = Vel
\]

\[
V = F
\]

\[
L = M
\]

\[
R = D
\]

\[
\frac{1}{C} = K
\]
Lumped Element Modeling of Transducer

- Electro-mechanical transducer converts electrical signal (voltage) to mechanical signal (force)
- It is represented by a transformer with \( \eta:1 \) ratio
- Larger \( \eta \), higher efficiency (\( \eta \) usually lower than %10)

\[
\frac{F(t)}{V(t)} = \frac{\eta}{1}
\]

Transducer

\( V(t) \) \( \rightarrow \) \( F(t) \)

\( D_{eq} \) \( M_{eq} \) \( 1/K_{eq} \)